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PROBLEMS AND PROPERTIES OF A NEW DIFFERENTIAL OPERATOR

(Masalah dan Sifat-sifat suatu Pengoperasi Pembeza Baharu)

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ABSTRACT

In this paper, we introduce and study a new differential operator defined in the open unit disc $U=\{z:z\in C\wedge|z|<1\}$. Using this operator, we then introduce a new subclass of analytic functions $G_n(\mu,\lambda,\alpha,\beta,b)$. Moreover, we discuss coefficient estimates, growth and distortion theorems and inclusion properties for the functions belonging to the class $G_n(\mu,\lambda,\alpha,\beta,b)$.

Keywords: Analytic functions; convex functions; differential operator

ABSTRAK

Dalam makalah ini, pengoperasi pembeza baharu dalam cakera unit $U=\{z:z\in C\wedge|z|<1\}$ diperkenalkan dan dikaji. Dengan menggunakan pengoperasi ini, subkelas baru fungsi analisis $G_n(\mu,\lambda,\alpha,\beta,b)$ diperkenalkan. Malah anggaran pekali, teorem pertumbuhan dan erotan, dan sifat rangkuman untuk kelas $G_n(\mu,\lambda,\alpha,\beta,b)$ turut dibincangkan.

Kata kunci: Fungsi analisis; fungsi cembung; pengoperasi pembeza

1. Introduction and Preliminaries

Let A denote the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k, \tag{1}$$

which are analytic and normalised (in usual sense) in the open unit disc $U=\{z:z\in C\wedge|z|<1\}$. For a function f in A, we define the following differential operator:

$$D^0 f(z)=f(z); \tag{2}$$

$$D^1 f(z)=\left(\frac{\alpha-\beta-\lambda}{\alpha}\right)f(z)+\left(\frac{\beta+\lambda}{\alpha}\right)zf'(z); \tag{3}$$

$$\vdots$$

$$D^n f(z)=D_\lambda(D^{n-1}f(z)). \tag{4}$$

If f is given by (1), then from (4) we get

$$D^n f(z) = z + \sum_{k=2}^{\infty} \left(\frac{\alpha + (\beta + \lambda)(k-1)}{\alpha}\right)^n a_k z^k \tag{5}$$

where $f \in A, n \in N_0$. This generalises many operators as follows.

(i) When $\alpha = 1, \beta = 0,$ we get

$$D^n f(z) = z + \sum_{k=2}^{\infty} (1 + \lambda(k-1))^n a_k z^k$$

the so-called Al-Oboudi (2004) differential operator.

(ii) When $\alpha = 1, \beta = 0$ and $\lambda = 1,$ we get

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k$$

the Sălăgean's (1983) differential operator.

(iii) When $\alpha = 2, \beta = 0$ and $\lambda = 1,$ we get

$$D^n f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+1}{2}\right)^n a_k z^k$$

a differential operator given by Uralegaddi and Somanatha (1992).

(iv) When $\beta = 1, \lambda = 0$ and replacing α by $\alpha + 1,$ we get

$$D^n f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+\alpha}{\alpha}\right)^n a_k z^k$$

the differential operator of Cho and Srivastava (2003).

(v) When $\beta = 0$ and replacing α by $\alpha + 1,$ we get

$$D^n f(z) = z + \sum_{k=2}^{\infty} \left(\frac{\alpha + \lambda(k-1) + 1}{\alpha + 1}\right)^n a_k z^k$$

a well known differential operator of Aouf *et al*. (2009).

Let $G_n(\mu,\lambda,\alpha,\beta,b)$ denote the subclass of A consisting of functions f which satisfy

$$\mathrm{Re}\{1+\frac{1}{b}[(1-\mu)\frac{D^n f(z)}{z}+\mu(D^n f(z))'-1]\}>0, \tag{6}$$

where $D^n f(z)$ is given by (5).

This implies that it satisfies the following inequality

$$\left|\frac{(1-\mu)\frac{D^n f(z)}{z}+\mu(D^n f(z))'-1}{(1-\mu)\frac{D^n f(z)}{z}+\mu(D^n f(z))'-1+2b}\right|<1 \tag{7}$$

where $z\in U;\mu\geq 0;n\in N_0;b\in C-\{0\}$.

We note that

(*i*) $G_0(\mu,\lambda,\alpha,\beta,b)=G(\lambda, b)$

$$\mathrm{Re}\{f\in A:\mathrm{Re}\{1+\frac{1}{b}[(1-\mu)\frac{f(z)}{z}+\mu(f(z))'-1]\}, z\in U\}>0,$$

(*ii*) $G_n(0,1,1,0,b)=G_n(b)$

$$\mathrm{Re}\{f\in A:\mathrm{Re}\{1+\frac{1}{b}[\frac{D^n f(z)}{z}-1]\}, z\in U\}>0,$$

(*iii*) $G_n(1,1,1,0,b)=R_n(b)$

$$\mathrm{Re}\{f\in A:\mathrm{Re}\{1+\frac{1}{b}[(D^n f(z))'-1]\}, z\in U\}>0,$$

(*iv*) $G_0(0,1,1,0,b)=G(b)$

$$\mathrm{Re}\{f\in A:\mathrm{Re}\{1+\frac{1}{b}[\frac{f(z)}{z}-1]\}, z\in U\}>0,$$

(*v*) $G_0(1,1,1,0,b)=R(b)$

$$\mathrm{Re}\{f\in A:\mathrm{Re}\{1+\frac{1}{b}[(f(z))'-1]\}, z\in U\}>0,$$

$(vi)\ G_0(0,1,1,0,1-\alpha) = G_\alpha$

$\mathrm{Re}\{f \in A : \mathrm{Re}\dfrac{f(z)}{z} > \alpha,\ 0 \le \alpha < 1,\ z \in U\} > 0,$

$(vii)\ G_0(1,1,1,0,1-\alpha) = R_\alpha$

$\mathrm{Re}\{f \in A : \mathrm{Re}(f(z))' > \alpha,\ 0 \le \alpha < 1,\ z \in U\} > 0 \cdot$

The class $R(b)$ was studied by Halim (1999), the class G_α by Chen (1974; 1975) and whereas the class R_α by Ezrohi (1965).

2. Coefficient Inequalities

In this section we find the coefficient inequality for the class $G_n(\mu,\lambda,\alpha,\beta,b)$.

Theorem 1. *Let the function* f *defined by* (1) *satisfies the condition*

$$\sum_{k=2}^{\infty}[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n\,|a_k| \le |b|\ (\mu \ge 0, n \in N_0, \alpha > 0, \beta+\lambda > 0) \cdot \tag{8}$$

Then $f \in G_n(\mu,\lambda,\alpha,\beta,b) \cdot$

Proof. Suppose that the inequality (8) holds. Then we have for $z \in U$

$$|(1-\mu)\frac{D^n f(z)}{z} + \mu(D^n f(z))' - 1| -$$

$$|(1-\mu)\frac{D^n f(z)}{z} + \mu(D^n f(z))' + 2b - 1|$$

$$= |\sum_{k=2}^{\infty}[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n\,|a_k|\,z^{k-1}| -$$

$$|2b + \sum_{k=2}^{\infty}[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n\,|a_k|\,z^{k-1}|$$

$$\le \sum_{k=2}^{\infty}[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n\,|a_k|\,|z^{k-1}| -$$

$$2|b|-\sum_{k=2}^{\infty}[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n|a_k||z^{k-1}|$$
$$\leq\{\sum_{k=2}^{\infty}[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n|a_k||z^{k-1}|$$
$$-|b|\}\leq 0.$$

where $D^n f(z)$ is given by (5).

This implies

$\sum_{k=2}^{\infty}[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n|a_k|\leq|b|,$ which shows that $f\in G_n(\mu,\lambda,\alpha,\beta,b)\cdot$

Corollary 1. *Let the function* f *defined by* (1) *be in the class* $G_n(\mu,\lambda,\alpha,\beta,b)\cdot$ *Then we have*

$$|a_k|\leq\frac{|b|}{[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n},\ k\geq 2.$$

Corollary 2. *Let the hypotheses of Theorem* 2.1 *be satisfied. Then for* $\beta=\lambda=0$ *and* $\mu=1$ *we have*

$$|a_k|\leq\frac{|b|}{k},\ k\geq 2.$$

3. Growth and Distortion Theorems

A growth and distortion property for function *f* to be in the class $G_n(\mu,\lambda,\alpha,\beta,b)$ is given as follows:

Theorem 2. *If the function f defined by* (1) *is in the class* $G_n(\mu,\lambda,\alpha,\beta,b)$, *then for* $|z|<1$, *we have*

$$|f(z)|\leq|r|+\frac{|b||r|^k}{[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n},$$
$$|f(z)|\geq|r|-\frac{|b||r|^k}{[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n}.$$

Proof. Let $f\in G_n(\mu,\lambda,\alpha,\beta,b),$ then by Theorem 1. We have

$$\sum_{k=2}^{\infty}[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^{n}\ |a_k|\leq|b|$$

$\Longrightarrow$

$$\sum_{k=2}^{\infty}|a_k|\leq\frac{|b|}{[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^{n}}.$$

From equation (1) we have

$$|f(z)|=\left|z+\sum_{k=2}^{\infty}a_k z^k\right|\leq|z|+\sum_{k=2}^{\infty}|a_k||z^k|.$$

Which implies

$$|f(z)|\leq|z|+\sum_{k=2}^{\infty}|a_k||z^k|,$$

$$|f(z)|\leq r+\frac{|b|}{[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^{n}}r^k.$$

Similarly we can prove that

$$|f(z)|\geq r-\frac{|b|}{[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^{n}}r^k.$$

Theorem 3. *Let the hypotheses of Theorem* 1 *be satisfied, then for* $|z|<1$,

$$|f(z)|\leq|r|+\frac{\alpha^n|b||r|^2}{[1+\mu][\alpha+\beta+\lambda]^n}$$

$$|f(z)|\geq|r|-\frac{\alpha^n|b||r|^2}{[1+\mu][\alpha+\beta+\lambda]^n}.$$

Proof. From Theorem 1 we have $f\in G_n(\mu,\lambda,\alpha,\beta,b)$ and hence

$$\sum_{k=2}^{\infty}[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^{n}\ |a_k|\leq|b|.$$

Since

$$[1+\mu][\frac{\alpha+\beta+\lambda}{\alpha}]^n\sum_{k=2}^{\infty}|a_k| \le \sum_{k=2}^{\infty}[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n |a_k| \le |b|$$

we have

$$[1+\mu][\frac{\alpha+\beta+\lambda}{\alpha}]^n\sum_{k=2}^{\infty}|a_k| \le |b|.$$

From (1) we have

$$|f(z)| \le |z| + \sum_{k=2}^{\infty}|a_k||z^k| \le |z| + \sum_{k=2}^{\infty}|a_k||z^2|,$$

$$|f(z)| \le |z| + \sum_{k=2}^{\infty}|a_k||z^2|.$$

Which proves that

$$|f(z)| \le |r| + \frac{\alpha^n|b||r|^2}{[1+\mu][\alpha+\beta+\lambda]^n}.$$

Similarly

$$|f(z)| = \left|z + \sum_{k=2}^{\infty}a_k z^k\right| \ge |z| - \sum_{k=2}^{\infty}|a_k||z^k|$$

$$|f(z)| \ge |z| - \sum_{k=2}^{\infty}|a_k||z^2|$$

shows that

$$|f(z)| \ge |r| - \frac{\alpha^n|b||r|^2}{[1+\mu][\alpha+\beta+\lambda]^n}.$$

Corollary 3. *Let the hypotheses of Theorem* 1 *be satisfied, if* $\alpha = \lambda = \mu = 1,\ \beta = 0$ *then for* $|z| < 1,$ *we have*

$$|f(z)| \le |r| + \frac{|b||r|^2}{2^{n+1}}$$

$$|f(z)| \ge |r| - \frac{|b||r|^2}{2^{n+1}}.$$

Theorem 4. *If the function f defined by* (1) *is in the class* $G_n(\mu, \lambda, \alpha, \beta, b)$ *, then for* $|z| < 1,$ *we have*

$$|f'(z)| \le 1 + \frac{k|b||r|^{k-1}}{[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n}$$

$$|f'(z)| \ge 1 - \frac{k|b||r|^{k-1}}{[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n}.$$

Proof. Let $f \in G_n(\mu, \lambda, \alpha, \beta, b),$ then by using Theorem 1 we have

$$\sum_{k=2}^{\infty} |a_k| \le \frac{|b|}{[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n}.$$

Also

$$|f'(z)| = \left|1 + \sum_{k=2}^{\infty} k a_k z^{k-1}\right| \le 1 + k\sum_{k=2}^{\infty} |a_k||z^{k-1}|$$

$$|f'(z)| \le 1 + k\sum_{k=2}^{\infty} |a_k||r^{k-1}|.$$

This shows that

$$|f'(z)| \le 1 + \frac{k|b||r|^{k-1}}{[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n}.$$

Similarly we can prove that

$$|f'(z)| \geq 1 - \frac{k|b||r|^{k-1}}{[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n}.$$

3. Inclusion properties

The inclusion properties for the class $G_n(\mu,\lambda,\alpha,\beta,b)$ are given by the following theorem.

Theorem 5. *Let the hypotheses of Theorem* 1 *be satisfied. Then*

$$G_n(\mu_2,\lambda,\alpha,\beta,b) \subseteq G_n(\mu_1,\lambda,\alpha,\beta,b)$$
$$G_n(\mu,\lambda_2,\alpha,\beta,b) \subseteq G_n(\mu,\lambda_1,\alpha,\beta,b)$$
$$G_n(\mu,\lambda,\alpha_1,\beta,b) \subseteq G_n(\mu,\lambda,\alpha_2,\beta,b)$$
$$G_n(\mu,\lambda,\alpha,\beta_2,b) \subseteq G_n(\mu,\lambda,\alpha,\beta_1,b)$$

where

$$\alpha_2 \geq \alpha_1, \beta_2 \geq \beta_1, \mu_2 \geq \mu_1 \text{ and } \lambda_2 \geq \lambda_1 \cdot$$

Proof. Let $f \in G_n(\mu_2,\lambda,\alpha,\beta,b)$. Then by using Theorem 1 we have

$$\sum_{k=2}^{\infty}[1+\mu_2(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n |a_k| \leq |b|$$

if $\mu_2 \geq \mu_1$, implying that $1+\mu_2(k-1) \geq 1+\mu_1(k-1)$,

in such that

$$\sum_{k=2}^{\infty}(1+\mu_2(k-1))(\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha})^n \geq \sum_{k=2}^{\infty}(1+\mu_1(k-1))(\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha})^n.$$

This shows that

$$\sum_{k=2}^{\infty}[1+\mu_1(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n |a_k| \leq |b| \leq \sum_{k=2}^{\infty}[1+\mu_2(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n |a_k| \leq |b|$$

or

$$\sum_{k=2}^{\infty}[1+\mu_1(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n \mid a_k \mid \leq \mid b \mid.$$

Hence $f \in G_n(\mu_1,\lambda,\alpha,\beta,b)$, which shows that $G_n(\mu_2,\lambda,\alpha,\beta,b) \subseteq G_n(\mu_1,\lambda,\alpha,\beta,b)\cdot$

Similarly, let $f \in G_n(\mu,\lambda,\alpha_1,\beta,b)$, then by using Theorem 1 we have $\alpha_2 \geq \alpha_1$. This implies that

$$(1+\frac{(\beta+\lambda)(k-1)}{\alpha_1})^n \geq (1+\frac{(\beta+\lambda)(k-1)}{\alpha_2})^n,$$

$$\sum_{k=2}^{\infty}(1+\mu(k-1))(\frac{\alpha_1+(\beta+\lambda)(k-1)}{\alpha_1})^n \geq \sum_{k=2}^{\infty}(1+\mu(k-1))(\frac{\alpha_2+(\beta+\lambda)(k-1)}{\alpha_2})^n$$

$$\sum_{k=2}^{\infty}[1+\mu(k-1)][\frac{\alpha_1+(\beta+\lambda)(k-1)}{\alpha_1}]^k \left|a_k\right| \leq \left|b\right|$$

and hence

$$\sum_{k=2}^{\infty}(1+\mu(k-1))(\frac{\alpha_2+(\beta+\lambda)(k-1)}{\alpha_2})^n \leq \left|b\right|.$$

This proves that $f \in G_n(\mu,\lambda,\alpha_2,\beta,b)$, and finally implies that

$$G_n(\mu,\lambda,\alpha_1,\beta,b) \subseteq G_n(\mu,\lambda,\alpha_2,\beta,b).$$

Employing a similar procedure we can prove that

$$G_n(\mu,\lambda_2,\alpha,\beta,b) \subseteq G_n(\mu,\lambda_1,\alpha,\beta,b)$$

and

$$G_n(\mu,\lambda,\alpha,\beta_2,b) \subseteq G_n(\mu,\lambda,\alpha,\beta_1,b)\cdot$$

For more details about coefficient bounds we refer to Joshi (2007), Aouf (1987), Silverman (1975), Raina (1997), and Owa and Aouf (1989), respectively.

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